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SERVER-VACATION MODELS

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TECHNICAL REPORT

Office of Naval Research

Contract No. N00014-86-K-0029

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Report No. GMU/49146/102

March 1986

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Copy No. *19*

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD A 165 910	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) State Dependence in M/G/1 Server-Vacation Models		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER GMU/49146/102
7. AUTHOR(s) Carl M. Harris William G. Marchal		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0029
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Systems Engineering George Mason University Fairfax, VA 22901		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Task H-A Project 4118150
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 North Quincy Street Arlington, VA 22217		12. REPORT DATE March 1986
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE Unclassified

16. DISTRIBUTION STATEMENT (of this Report)

Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

computational probability	computer performance evaluation
computer networks	queueing models
local area networks	server vacations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

Abstract

This paper examines a generalization of the exhaustive and one-at-a-time-discipline M/G/1 server vacation models. This alternative model is viewed as a state-dependent (non-vacation) M/G/1 queue in which the original service times are extended to include a (possibly zero length) state-dependent vacation after each service. Such a vacation policy permits greater flexibility in modeling real problems, and does, in fact, subsume most prior M/G/1 approaches. This device reveals a fundamental decomposition somewhat like that previously established for the classical vacation disciplines. In addition, necessary and sufficient conditions for system ergodicity are established for the state-dependent vacation policy, and some comments are offered on computations together with a few illustrative examples. *Keywords: computer networks; local area networks.*
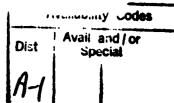
I. Introduction

The class of M/G/1 queueing models with a server who periodically goes on "vacation" is frequently offered as a tool to understand congestion phenomenon in local area networks. A central or common server appears to the user to disappear or "go on vacation" whenever the central server performs background or service which is alternative relative to the observer. This model applies as well to manufacturing processes which exhibit uninterruptible maintenance tasks such as tool changes or alterations of a flexible manufacturing system. Service processes subject to periodic breakdowns and interruptions are further examples of potential application.

Most vacation models of this type exhibit an interesting decomposition property. The number of customers in the system in the steady state can be interpreted as the sum of the state of a corresponding model with no vacations and a second nonnegative discrete random variable. In other words, the vacation model's limiting state distribution can be found by convolving the distribution in the non-vacation system with a second distribution. Sometimes one can interpret this second variable as the number of arrivals during the residual of a vacation period. A corresponding decomposition result occurs for the waiting time distribution as well.

Gaver [1962] first noted this decomposition. A few of the generalizations were presented in Cooper [1970], Levy and Yechiali [1975], Scholl and Kleinrock [1983], Fuhrmann [1984], and Fuhrmann and Cooper [1985]. Cases of G/G/1 vacation models were developed in Doshi [1985] and Keilson and Servi [1986]. The latter cases are more difficult

as size at the departure epochs are then not necessarily a Markov chain as is typical of the M/G/1 models. A complete survey of all of these works may be found in Doshi [1986].

II. Definitions and Other Preliminaries

In this paper, we study the M/G/1 queue with server vacations whose distributions can be considered state dependent. The flexibility afforded by the introduction of state-dependent vacation distributions allows us a more general approach than previous M/G/1 vacation models, which may provide a particularly useful perspective in their design and control.

A major element in our work is the formulation of this problem as a departure-point, state-dependent-service queue, in the sense of Harris [1967], and as also discussed in subsequent papers, like Harris [1969], and in Gross and Harris [1985]. These models assume that customer service-time distributions are indexed on the state of the system at service initiation. The Markovian character of the departure points of the regular M/G/1 is preserved, but the service times of successive customers need no longer be identically distributed. We focus on the stationary system-size probabilities, rather than the waiting times, since the former come more naturally out of the imbedded chain and there is the easy derivation of the one from the other in the M/G/1. Note that the general-time probabilities are equal to the imbedded departure-point probabilities here just as in the usual M/G/1.

In this work, then, we suppose that customers arrive to the system as a Poisson process with rate λ and have independent, identical service

times with cumulative distribution function B(t). These service times are also independent of the arrival process and of the vacation lengths. The line discipline is assumed not to depend upon service times, and vacations commence only at the completion of a service. In other words, no customer's service is interrupted or preempted.

Let $V_n(t)$ represent the cumulative distribution function for the length of a vacation which begins immediately after a service completion when there are n customers present. The specific quantity $V_n(0)$ denotes the probability that the server will <u>not</u> take a vacation given there are n in the system. We define $V_0(0) = 0$, while $V_n(0)$ could be non-zero for any n > 0. We shall use $\bar{v}_i$ to denote the mean length of a type i vacation.

By taking $V_0(t) = V(t)$ ($\neq 0$) and $V_n(0) = 1$ for all other n, we get what is called the "exhaustive server-vacation policy," that is, the server vacates for a random time only when all present customers have been served and the system has gone idle. On the other hand, when $V_i(t) = V_j(t) = V(t)$ for all i and j, we have the classical "one-at-a-time" server-vacation policy. All other "Markovian" server-vacation policies, which depend only on the state of the system at a customer departure point, can be represented by an appropriate choice of $\{V_n(t)\}$. Furthermore, the state space could be supplemented to model even more complex behavior with some kind of "finite" memory. We note that the Bernoulli switch vacation discipline of Keilson and Servi [1986] is a special type of "one-at-a-time" policy where V(t) has a jump at the origin.

Since the state-dependent vacation model is defined to permit the server (with probability $1 - V_n(0)$) to take leave after each service, our

model is a variation of the one-at-a-time vacation policy. Each service of the classical M/G/1 is replaced by a cycle which consists of one service and one (possibly zero length) vacation, with an allowance for repeated vacations when the server finds the queue still empty upon return. The transition matrix A of the state-dependent-vacation Markov chain imbedded at customer departure points has the structure

$$
A = \begin{bmatrix}
a_{00} & a_{10} & a_{20} & a_{30} & \cdot & \cdot & \cdot \\
a_{01} & a_{11} & a_{21} & a_{31} & \cdot & \cdot & \cdot \\
0 & a_{02} & a_{12} & a_{22} & \cdot & \cdot & \cdot \\
0 & 0 & a_{03} & a_{13} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \quad \cdot
$$

For $j > 0$,

$$
a_{ij} = \int_{0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{i}}{i!} \, d[B*V_{j}](t) = \Pr\{i \text{ arrivals occur during a type } j \text{ cycle}\}.
$$

where $B*V_{j}$ denotes the distribution which is the convolution of $B(t)$ and $V_{j}(t)$.

A unique cycle begins from a completely empty system. We have denoted this as a type 0 (zero) cycle. When the server begins this type of vacation, there are no customers present and it is conceivable that he might return before any have arrived. In that case, it is generally assumed that he takes another vacation which follows the same probability distribution, namely $V_{0}(t)$. Because there are no departures when this

occurs, the imbedded Markov chain is unaffected. However, the a_{io} probabilities are implicitly conditioned on the fact that at least one arrival has occurred during the vacation. Actually, a_{io} is the probability that $(i + 1)$ arrivals occur during a cycle, given at least one arrival occurs during the vacation period. Otherwise, the server would not have terminated his vacation. Thus

$$a_{io} = \frac{\int_o^\infty \frac{e^{-\lambda t}(\lambda t)^{i+1}}{(i + 1)!} d[B*V_o](t) - \int_o^\infty e^{\lambda t} dV_o(t) \cdot \int_o^\infty \frac{e^{-\lambda t}(\lambda t)^{i+1}}{(i + 1)!} dB(t)}{1 - \int_o^\infty e^{-\lambda t} dV_o(t)} \tag{1}$$

The first term in the numerator of (1) measures the likelihood of $(i + 1)$ arrivals during a cycle. The second term precludes the possibility that all the arrivals could occur during a fictitious sevice time. Finally, the denominator guarantees that at least one arrival has occurred during the vacation.

III. Central Results

In the following, we represent the stationary distribution of the M/G/1 state-dependent vacation queue by the vector $\pi = (\pi_o, \pi_1, \pi_2, \cdots)$ and its generating function as $\Pi(z)$. We shall write the generating function associated with the row probability vectors of the transition matrix A as

$$K_i(z) = \sum_{j=0}^\infty a_{ji} z^j .$$

The expected value $\left(\sum_{j=1}^{\infty} j a_{ji} \right)$ of the $i^{\underline{th}}$ row distribution is defined as ρ_i, $i = 0,1,2,\cdots$. It is then easy to show that

$$\rho_i \equiv E[\text{arrivals during a type } i \text{ cycle}]$$
$$= \lambda \left(\frac{1}{\mu} + \bar{v}_i \right) \qquad (i > 0) \quad,$$

and

$$\rho_o = \rho - \frac{\bar{v}_o - 1}{P(0)} \,,$$

where

$$P(0) = \int_0^{\infty} e^{-\lambda t} dV_o(t)$$

is the conditional probability of no arrivals during an arbitrary type 0 vacation. We shall retain the usual convention for $\rho = \lambda/\mu$ as the traffic intensity of the M/G/1 without vacation.

For later reference, we note the Pollaczek-Khintchine formula for the steady-state system-size generating function

$$\Pi(z) = \frac{\pi_o (1 - z) K(z)}{K(z) - z} \,.$$

Crabill [1968] demonstrated that a sufficient condition for ergodicity of the M/G/1 queue with state-dependent service times is $\limsup_i \{\rho_i\} < 1$. In this application this is equivalent to requiring that $\limsup_i \{\lambda(\frac{1}{\mu} + \bar{v}_i)\} < 1$, or $\limsup_i \{\lambda \bar{v}_i\} < 1 - \rho$. In other words, only a finite number of the state-dependent vacations can have a mean greater than $(1 - \rho)/\lambda$.

A necessary condition for ergodicity is the somewhat weaker requirement that $\liminf_i \{\rho_i\} < 1$. This follows from Theorem 4 of

Sennott et al. [1983] when you recognize that $\rho_i - 1$ is the mean "drift" of the Markov chain from state i and you reason from contraposition. In words, for every state of the system i, an infinite number of states $j > i$ must have the property that $\overline{v}_j < (1 - \rho)/\lambda$.

These conditions become important when there is operator discretion, for example, to alter vacations with state size, in an attempt to use the server effectively elsewhere, as in a local area network. But one cannot let the server stay away very long when system sizes are growing, for then ergodicity might be violated.

Now, under the assumption that a stationary distribution for the system size does indeed exist, a standard generating function argument (see Harris [1967]) leads to the relation

$$\Pi(z) = \pi_0 K_0(z) + \sum_{i=1}^{\infty} \pi_i z^{i-1} K_i(z). \tag{2}$$

(For a discussion of the derivation of the system-size probabilities in an analogous M/G/1 model, consult pages 289-290 of Gross and Harris [1985].)

The next observation is so important that we label it here as a theorem.

Theorem: If the probability generating functions, $\{K_i(z)\}$, for the rows $(i > 0)$ of the departure-point Markov chain of a state-dependent M/G/1 queue can each be expressed as a product of two generating functions, one of which is common to all rows, such that $K_i(z) = K(z) \cdot D_i(z)$, and if there exists some j such that $D_i(z) = D(z)$ for all $i > j$, then the stationary system size decomposes into the sum of the stationary system

size for a non-vacation variant of M/G/1 with a second discrete non-negative random variable.

<u>Proof</u>: Since $K_i(z) = K(z) \cdot D_i(z)$ for all i, it follows from (2) that

$$\Pi(z) = \pi_0 K(z) D_0(z) + \sum_{i=1}^{j-1} \pi_i z^{i-1} K(z) D_i(z)$$

$$+ \sum_{i=j}^{\infty} \pi_i z^{i-1} K(z) D(z)$$

$$= K(z) \left[\pi_0 D_0(z) + \frac{1}{z} \sum_{i=1}^{j-1} \pi_i z^i D_i(z) \right.$$

$$\left. + \frac{D(z)}{z} \left(\Pi(z) - \sum_{i=0}^{j-1} \pi_i z^i \right) \right].$$

Combining and simplifying gives

$$[z - D(z)K(z)]\Pi(z)$$

$$= K(z) \left[\pi_0 z D_0(z) + \sum_{i=1}^{j-1} \pi_i z^i D_i(z) \right.$$

$$\left. - D(z) \sum_{i=0}^{j-1} \pi_i z^i \right],$$

or

$$\Pi(z) = \frac{K(z)\left\{\pi_0[zD_0(z) - D(z)] + \sum_{i=1}^{j-1} \pi_i z^i [D_i(z) - D(z)]\right\}}{z - D(z)\,K(z)}$$

$$= \frac{K(z)(z - 1)\,\hat{\pi}_0}{z - D(z)K(z)}$$

$$X \frac{1}{\hat{\pi}_0} \frac{\pi_0[zD_0(z) - D(z)] + \sum_{i=1}^{j-1} \pi_i z^i [D_i(z) - D(z)]}{z - 1} . \qquad (3)$$

From (2), the first term of (3) is the generating function of a state-dependent M/G/1 where $\hat{\pi}_0$ is the equilibrium empty probability and K(z) is the generating function for the top row of the imbedded matrix, while the product D(z)K(z) is the generating function for all other rows. The second factor of (3) is the generating function for another counting process whose precise interpretation is more easily seen in a vacation format, as in the following.

Q.E.D.

Two special cases for (3) are of special interest. For the first special case (related to one-at-a-time service), let $D_i(z) = D(z)$ for all $i > 0$. Then

$$\Pi(z) = \frac{K(z)(z - 1)\hat{\pi}_0}{z - D(z)K(z)}$$

$$X \frac{1}{\hat{\pi}_0} \left\{\frac{\pi_0[zD_0(z) - D(z)]}{z - 1}\right\} .$$

The second (related to exhaustive service) requires $D(z) = 1$. Then

$$\Pi(z) = \frac{K(z)(z-1)\,\hat{\pi}_o}{z - K(z)}\; \frac{\pi_o}{\hat{\pi}_o}\; \frac{\left[zD_o(z) - 1\right]}{z - 1}\;.$$

To apply the theorem to the general, state-dependent vacation model, we first define $C_i(z)$ to be the probability generating function for the number of arrivals during a type i (>0) vacation, with $C_i(z) = C(z)$ for all $i > j$ and $C_0(z)$ defined as the (conditional) generating function for arrivals during the "final" vacation after idleness. In addition, let $F_0(z)$ be the (unconditional) generating function for the number of arrivals during an arbitrary type 0 vacation. Since, for $i > 0$, $K_i(z)$ is the generating function of the number of arrivals during a cycle of type i created by the sum of a service time and a type of i vacation, it follows that

$$K_i(z) = K(z) \cdot C_i(z).$$

For the case $i = 0$, we observe from (1) that

$$z[1 - P(0)]K_o(z) = z[1 - P(0)] \sum_{i=0}^{\infty} a_{io} z^i$$

$$= \sum_{i=0}^{\infty} \int_{o}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{i+1} z^{i+1} d[B*V_o](t)}{(i + 1)!}$$

$$- P(0) \sum_{i=0}^{\infty} \int_{o}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{i+1} z^{i+1} dB(t)}{(i + 1)!}$$

$$= K(z)F_o(z) - P(0)K(z)$$

$$= K(z)[F_o(z) - P(0)].$$

Thus

$$K_0(z) = \frac{F_0(z) - P(0)}{z[1 - P(0)]} K(z) \ .$$

But it is easy to show that

$$C_0(z) = \frac{F_0(z) - P(0)}{1 - P(0)} \ ,$$

and therefore

$$K_0(z) = \frac{C_0(z)}{z} K(z).$$

Thus we see that all $\{K_i(z)\}$ can be written as products, and thus

$$D_i(z) = \begin{cases} C(z) & (i > j) \\ C_i(z) & (0 < i < j) \\ \dfrac{C_0(z)}{z} & (i = 0) \end{cases} \ .$$

(Note that the classical one-at-a-time discipline has $C_i(z) = C(z)$ for all $i > 0$ and $F_0(z) = C(z)$, while the exhaustive discipline requires that $C_i(z) = 1$ for $i > 0$.)

The application of the theorem and Equation (3) for the state-dependent vacation model then gives our fundamental decomposition result:

$$\Pi(z) = \frac{K(z)(z - 1) \, \hat{\pi}_0}{z - C(z)K(z)}$$

$$X \frac{1}{\hat{\pi}_0} \left\{ \frac{\pi_0 \, [C_0(z) - C(z)] + \sum_{i=1}^{j-1} \pi_i z^i \, [C_i(z) - C(z)]}{z - 1} \right\} \qquad (4)$$

where $\hat{\pi}_o$ is the stationary probability that the non-vacation M/G/1 is empty.

Clearly, the state-dependent model can be specialized to either the usual one-at-a-time or exhaustive disciplines. As noted, the one-at-a-time approach has $C_i(z) = C(z)$ for all $i > 0$. Thus (4) simplfies to

$$\Pi(z) = \frac{K(z)(z-1)\,\hat{\pi}_o}{z - C(z)K(z)}$$

$$X \,\frac{1}{\hat{\pi}_o}\;\;\frac{\pi_o\left[\frac{F_o(z) - P(0)}{1 - P(0)} - C(z)\right]}{z - 1}\;.$$

The first factor of this product corresponds to a simple state-dependent M/G/1 as in Equation (3). The second factor can be re-written as

$$\frac{\pi_o}{\hat{\pi}_o}\cdot\frac{P(0)}{1 - P(0)}\cdot\frac{1 - C(z)}{1 - z}\quad[\text{because } F_o(z) = C(z)].$$

Then one can recognize the probability generating function for the number of arrivals during a residual vacation, namely,

$$\frac{1 - C(z)}{\lambda\bar{v}(1-z)}\;.$$

When $C_i(z) = 1$ for $i > 0$, we get the exhaustive discipline. Then, (4) simplifies to

$$\Pi(z) = \frac{K(z)(z-1)\hat{\pi}_o}{z - K(z)}\;\frac{\pi_o}{\hat{\pi}_o}\;\frac{F_o(z) - 1}{[1 - P(0)](z - 1)}$$

$$= \frac{K(z)(z-1)\hat{\pi}_o}{z - K(z)} \frac{\pi_o}{\hat{\pi}_o} \frac{C_o(z) - 1}{z - 1}.$$

Now this is precisely the expected decomposition, as, for example, found in Cooper [1970]. The first term is the exact P-K result for the M/G/1, while the second term is the probability generating function for the number of arrivals in a residual "last" vacation time before the start of the busy period.

A more complete interpretation of the decomposition in Equation (4) requires a careful analysis. The first factor is again the probability generating function for simple state-dependent M/G/1 whose top row has generating function $K(z)$, while all others have generating function $C(z)$ $K(z)$. This product function results from an effective service time equal to the sum of the actual service time and the vacation time.

The second factor of (4) can be rewritten as

$$\sum_{i=0}^{\infty} \frac{\pi_i z^i C_i(z)}{z - 1} - \sum_{i=0}^{\infty} \frac{\pi_i z^i C(z)}{z - 1} \tag{5}$$

since $C_i(z) = C(z)$ for all $i > j$. This is the difference between two probability generating functions. The first term of (5) is for the number of customers the server would see in <u>queue</u> at the very beginning of a fresh service just after returning from a state-dependent vacation. This is so because $C_i(z)/(z-1)$ is the generating function for the number of arrivals during a residual vacation of type i, that is, during the time left after the first arrival in vacation. (Note that $C_o(z) = C(z)$ implies that $C_o(z) = F_o(z)$, which is true if and only if $C(z) \equiv 1$ and there are no vacations.)

Now, the second term of (5) is the generating function product $\Pi(z) \cdot C(z)/(z - 1)$, which corresponds to the steady-state departure size plus the number of arrivals during a residual vacation taken according to the asymptotic distribution $V(t)$, that is, as if the state dependence had not occurred. This corresponds to the number in queue at the start of a service in a system with no state dependence. Thus we see that the difference of Equation (5) is a measure of the effect of the state dependence.

IV. Some Sample Problems and Comments on Computations

A few particular examples may illustrate the theoretical results and the necessary computations. First, in the non-state-dependent "one-at-a-time" case with exponential service and vacations with rates μ and v, respectively, we have from Equation (4) that

$$\Pi(z) = \frac{\mu v(1 - z)\pi_o}{\mu v - z[\mu + \lambda(1 - z)][v + \lambda(1 - z)]}$$

$$= \frac{\mu v(1 - z)\pi_o}{\mu v - \lambda(\mu + v + \lambda)z + \lambda^2 z^2} \, .$$

To get π_o, we recognize that $\Pi(1) = 1$ and thus

$$\frac{\mu v \pi_o}{\mu v - \lambda(\mu + v)} = 1$$

or

$$\pi_o = \frac{\mu v - \lambda(\mu + v)}{\mu v} = 1 - \frac{\lambda}{\mu} - \frac{\lambda}{v} \, .$$

Therefore

$$\Pi(z) = \frac{\mu v - \lambda(\mu + v)}{\mu v - \lambda(\mu + v + \lambda)z + \lambda^2 z^2} .$$

From this we can compute the mean system size as

$$L = \Pi'(1) = \frac{\lambda(\mu + v - \lambda)}{\mu v - \lambda(\mu + v)} = \frac{\frac{\lambda}{\mu}\left(1 + \frac{\mu}{v} - \frac{\lambda}{v}\right)}{1 - \frac{\lambda}{\mu} - \frac{\lambda}{v}} .$$

Now if $W*(s)$ denotes the Laplace transform of the sojourn distribution, we know that

$$W*(s) = \Pi\left(1 - \frac{s}{\lambda}\right) = \frac{\mu v - \lambda(\mu + v)}{s^2 + (\mu + v - \lambda)s + \mu v - \lambda(\mu + v)} .$$

Let s_1 and s_2 ($\neq s_1$) be the denominator's roots. Then the transform has the product form

$$W*(s) = \frac{\mu v - \lambda(\mu + v)}{(s - s_1)(s - s_2)}$$

$$= \frac{\mu v - \lambda(\mu + v)}{s_1 - s_2}\left(\frac{1}{s - s_1} - \frac{1}{s - s_2}\right) .$$

So the sojourn-time density is

$$w(t) = \frac{\mu v - \lambda(\mu + v)}{s_1 - s_2}\left(e^{s_1 t} - e^{s_2 t}\right) .$$

Suppose $\mu = 1/2$, $v = 1$, and $\lambda = 1/4$. Then it follows that $\pi_o = 1/4$ and $L = 5/2$. The appropriate quadratic is $s^2 + 5s/4 + 1/8$, with roots $s_1 = (\sqrt{17} - 5)/8 = -0.109612$ and $s_2 = (-\sqrt{17} - 5)/8 = -1.140388$, so that

$s_1 - s_2 = \sqrt{17}/4$. Hence the system waiting-time distribution is the simple generalized Erlang given by

$$w(t) = \frac{1}{2\sqrt{17}} \left(e^{s_1 t} - e^{s_2 t} \right) = 0.121268 \left(e^{-0.109612t} - e^{-1.1403884t} \right).$$

As a second example, consider the particular state-dependent case for which the vacation is different (say at rate v_0) when the server leaves from an empty system. As before, suppose the service and both vacation distributions are exponential. Equation (4) leads to the slightly more complex generating function expression

$$\Pi(z) = \frac{v_0 [vv_0 - \lambda(v_0 z - v)][\mu v - \lambda(\mu + v)]}{[v_0 + \lambda(1 - z)][vv_0 - \lambda(v_0 - v)][\mu v - \lambda(\mu + v + \lambda)z + \lambda^2 z^2)]}.$$

Now let $\mu = 1/2$, $v = 1$, $\lambda = 1/4$ and $v_0 = 1/2$. Then

$$\Pi(z) = \frac{4(6 - z)}{5(24 - 29z + 10z^2 - z^3)}.$$

We see that $\pi_0 = 1/5$ and $L = 14/5$. In contrast to the first example, the longer vacation in the "zero case" decreases the proportion of time that the server leaves an empty system and increases the average system size from 2.5 to 2.8. Here, the waiting-time transform is

$$W^*(s) = \frac{2s/5 + 1/2}{(s + 1/2)(s - s_1)(s - s_2)},$$

where $s_1 = (\sqrt{17} - 5)/8$ and $s_2 = (-\sqrt{17} - 5)/8$.

A partial fraction expansion is again available to give

$$W^*(s) = \frac{A}{s + 1/2} + \frac{B}{s - s_1} + \frac{C}{s - s_2} ,$$

where

$$A = \frac{-6}{5} , \quad B = \frac{51 + 11\sqrt{17}}{85} = 1.133578, \text{ and } C = \frac{51 - 11\sqrt{17}}{85} = 0.066422.$$

Finally,

$$w(t) = 1.133578e^{-0.109612t} + 0.06642e^{-1.140388t}$$
$$-1.2e^{-0.5t},$$

which is seen to be a three-term generalized Erlang distribution. As expected, we note that the first two exponential scale parameters are the same as encountered in the previous example.

In the exhaustive-service version of these problems, the generating function is

$$\Pi(z) = \frac{K(z)(z - 1)\pi_0}{z - K(z)} \frac{C_0(z) - 1}{z - 1} ,$$

with

$$K(z) = \frac{\mu}{\mu + \lambda(1 - z)}$$

and

$$C_0(z) = \frac{v_0 z}{v_0 + \lambda(1 - z)} .$$

Thus

$$\Pi(z) = \frac{\mu \pi_0 (v_0 + \lambda)}{(\mu - \lambda z)[v_0 + \lambda(1 - z)]} .$$

Since

$$\Pi(1) = 1 = \frac{\mu(v_0 + \lambda)\pi_0}{(\mu - \lambda)v_0} ,$$

it follows that

$$\pi_o = \frac{(\mu - \lambda)v_o}{\mu(v_o + \lambda)} = \frac{1}{3} \quad \text{and} \quad \Pi(z) = \frac{\mu - \lambda}{\mu - \lambda z} \cdot \frac{v_o}{v_o + \lambda(1 - z)} \,.$$

The first factor of this is proportional to the generating function of an ordinary M/G/1, while the second is $[C_o(z)/z]$.

For the waiting-time distribution of this exhaustive-service problem, the decomposition is immediate, giving

$$W^*(s) = \Pi\left(\frac{\lambda - s}{\lambda}\right) = \frac{\mu - \lambda}{\mu - \lambda + s} \cdot \frac{v_o}{v_o + s}$$

and

$$w(t) = \frac{(\mu - \lambda)v_o}{v_o - \mu + \lambda} \left\{ e^{-(\mu - \lambda)t} - e^{-v_o t} \right\}$$

$$= \frac{1}{2} \left\{ e^{-t/4} - e^{-t/2} \right\}$$

since $v_o = 1/2$, $\lambda = 1/4$, and $\mu = 1/2$.

These sample computations were particularly easy because the probability generating functions were low-order rational functions (specifically, with quadratic and cubic denominators). In the general case, it is apparent from Equation (4) that if the service and vacation distributions lead to rational functions (with possibly complex poles) for $K(z)$ and the $\{C_i(z)\}$, the system-size generating function $\Pi(z)$ will also be rational. Clearly, the rationality of the service distribution and the vacation distributions is sufficient for this to occur. The order of the denominator polynomial of $\Pi(z)$ relates directly to the degrees of the denominators in $K(z)$ and the $\{C_i(z)\}$. Likewise, the

sojourn time in system has a rational transform with order no greater
than the degree of the denominator polynomial in $\Pi(z)$. Similar
statements can be made to allow the system waiting times to have phase-
type distributions. The closure of phase types under convolutions and
finite mixtures (as documented in Neuts [1981]) gives a totally parallel
series of sufficient conditions for $w(t)$ to be phase.

V. Concluding Remarks

The central insight of this study is that the state-dependent model
is an efficient way to consolidate server vacation models with
exhaustive, "one-at-a-time", Bernoulli, and related vacation disciplines
into a single comprehensive model. In fact, the state-dependent M/G/1
server-vacation models are a subclass of the state-dependent-service
M/G/1 models. Hence classical Markov-chain analyses apply with only a
moderate amount of additional computational effort.

Possible future extensions of this work include an examination of the
existence and computation of an optimal control policy selected from a
menu of vacation policies under some cost structure. Proper formulation
of the vacation model as a state-dependent-service M/G/1 is expected to
lead to conditions which are necessary for the existence and calculation
of a vacation policy which is a function of the queue-length distribution
at the imbedded departure points. The computation of optimal policies
for non-state-dependent M/G/1 queues has already been addressed in this
open literature.

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